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Cylindrically Symmetric Perfect Fluid with Spin

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Abstract*— *The Present Paper Provides Solutions of E-C Field equations for static cylindrically symmetric perfect fluid by choosing a suitable equations of state and by using a specific form for one of the metric potentials we have assumed. The spins of all the individual particles composing the fluid to be aligned along the symmetry axis. The constants appearing in the solution have been found by using boundary condiyions, pressure, and density have been also found for the distribution such type of investigation is very useful for developments in relativistic astrophysics.

Index Terms*— *Perfect fluid, Metric potential, Spin, Pressure, and Density.

I. INTRODUCTION

The modified version of Einstein's theory of Gravitation, called Einstein-Cartan theory has been extensively studied by Hehl et al [1] Prasanna [7] has considered a study of static fluid sphere in E-C theory as an attempt to investigate whether Einstein-Cartan theory admits self-gravitating fluid systems .static fluid sphere in E-C Theory has been also studied by Kerlick [4], Kuchowicz [5] Skinner and Webb [10] Sah and Chandra [11] and Singh and Yadav [9] whereas Yadav et.al.[12] have studied non –static perfect fluid spheres with conformal flatness and also static fluid sphere of charge and uncharged cases.

However in spherical symmetry it is assumed that spins are aligned in radial direction (implying the presence of magnetic monopole at the center), the picture is not very physical. Further as a rotating system cannot be spherical, naturally it seems desirable to study axis symmetric distributions which are more physical keeping this fact in mind Prasanna [8] has considerd the simplest axis symmetric system namely a static cylinder of perfect fluid in Einstein-Cartan theory.

Here in this paper we have also studied a static cylindrically symmetric perfect fluid with spin and have solved the field equations using a suitable equation of state *viz.* $\bar{\rho}=a p_r$ and by choosing specific form for one of the metric potentials. We have assumed the spins of all the individual particles composing the fluid to aliged along the symmetry axis. The constants appearing in the solution have been found by boundry conditions pressure and density have also evaluated for the distribution.

II. THE FIELD EQUATIONS

We consider the static cylindrically symmetric metric given by

(2.1) $$ds^2 = - e^{2\alpha-2\beta}(dr^2 + dz^2) - r^2 e^{-2\beta} d\Phi^2 + e^{2\beta}dt^2$$

Where α and β are functions of 'r' alone.

Considering the perfect fluid material distribution with anistropic pressure the symmetric energy momentum tensor $T^{-I}{}_J$ is given by

(2.2) $$T^{-I}{}_J = diag.\{-p_r, -p_\Phi, -p_z, \rho\}$$

The Einstein –Cartan field equations are

(2.3) $$G^I{}_J \equiv R^I{}_J - \frac{1}{2} R\delta^I{}_J = - Kt^i{}_j,$$

(2.4) $$Q^I{}_{JK} - \delta^I{}_J Q^l{}_{lk} - \delta^i{}_k Q^l{}_{jl} = -KS^i{}_{jk}$$

Where R_{IJ} is Ricci Tensor, R is Scalar of curvature and t^i_j is canonical asymmetric energy momentum tensor.

The Canonical asymmetric energy momentum tensor is given by

(2.5) $$t^i_j = T^{-i}{}_j + \frac{1}{2} g^{im} \Delta KS^k{}_{jm}$$

The non-zero components of the Canonical tensor $t^i{}_j$ are

(2.6) $$t^1{}_1 = T^{-1}{}_1 = -P_r,\ t^2{}_2 = T^{-2}{}_2 = -P_\varphi,\ t^3{}_3 = T^{-3}{}_3 = -p_z,\ t^4{}_4 = T^{-4}{}_4 = \rho,\ t^4{}_2 = \frac{1}{2} K e^{\beta-\alpha}\beta^{1,}\ t^2{}_4 = \frac{1}{2} K e^{\beta-\alpha}\beta^1$$

Using equations *(2.3)* and *(2.6)* the field equations may be written as (Prasanna [7, 8])

(2.7) $$K e^{2(\beta-\alpha)}(2\beta'' - \alpha'' + \frac{2}{\gamma}\beta' - \beta'^2) + \frac{1}{4} K^2.K^2 = - K\rho$$

(2.8) $$e^{2(\beta-\alpha)} (\beta'^2 - + \frac{\alpha'}{\gamma}) - \frac{1}{4} K^2.\ K^2 = K\rho_r$$

(2.9) $$e^{2(\beta-\alpha)} (- \alpha^{11} - \beta^{12}) - \frac{1}{4} K^2.K^2 = KP_\varphi$$

(2.10 $$e^{2(\beta-\alpha)} (- \beta^{12} + \frac{\alpha'}{\gamma}) - \frac{1}{4} K^2.\ K^2 = KP_Z$$

(2.11) $$e^{2(\beta-\alpha)} (K^1 + K\alpha - K\beta^1) = - K e^{(\beta-\alpha)} \beta^1\ 9$$

(2.12) $$e^{(\beta-\alpha)} (K^1 + K\alpha^1 - K\beta^1) = K e^{(\beta-\alpha)} \beta^1$$

Adding (2.11) and (2.12), we get

(2.13) $K^1 + K\alpha^1 = 0$ which on integration gives

(2.14) $K = H e^{-\alpha}$, Where H is an arbitrary constant to be determined.

Following Hehl's approach [2, 3] by redefining pressure and density as

(2.15) $$P^- = P - 2\pi k^2, \rho^- = \rho - 2\pi k^2,$$

The field equations finally reduce to

(2.16) $$8\pi \rho^- = e^{2(\beta-\alpha)} (2\beta^{11} - \alpha^{11} c - \beta^{12})$$

(2.17) $$8\pi P^-{}_r = - 8\pi P^-{}_z = e^{2(\beta-\alpha)} (\frac{\alpha'}{\gamma} - \beta^{12})$$

(2.18) $$8\pi P^-{}_\varphi = e^{2(\beta-\alpha)} (\alpha^{11} + \beta^{12})$$

Also the continuity equation becomes

(2.19) $$\frac{d P^-{}_r}{d\gamma} + (\rho^- + P^-{}_r)\beta - (P^-{}_r - P^-{}_\varphi)(\beta^1 - \frac{1}{\gamma}) - 2 P^-{}_r(\beta^1 - \alpha^1) = 0$$

III. SOLUTION OF THE FIELD EQUATIONS

We have only three independent equations to determine five unknowns. Thus the system is indeterminate for complete determinacy of the system we require two more conditions. for this we assume an equation of state of the form

(3.1) $\bar{\rho} = a \bar{P}_r$

Where *a* is a constant. This gives us an additional equation.

(3.2) $2\beta^{11} + 2\frac{\beta'}{\gamma} - (1\text{-}\alpha)\beta'^2 = \frac{\alpha\alpha'}{\gamma} + \alpha^{11}$

Since our set of equation is still incomplete, we will take a judicious choice of one of the metric potentials for this we choose

(3.3) $\beta = B_1 r^2 + B_2$

Where B_1 and B_2 are constants with the value of 'β' equation *(3.2)* yields

(3.4) $\frac{d^2\alpha}{dr^2} + \frac{a}{r}\frac{d\alpha}{dr} = 8B_1 - 4(1-a)B_1^2 r^2$

Its solution is given by

(3.5) $\alpha = C_1\frac{r^{1-a}}{1-a} + 4B_1\frac{r^2}{1+a}\text{-}\ 1\text{-}a)\ \frac{r^4}{(a+3)}B_1^2 + C_2$

Where C_1 and C_2 are constants of integration.

We have now four arbitrary constants B_1, B_2, C_1, and C_2 which are to be determined by the boundary conditions. Assuming that the cylinder has a radius $r = r_0$ we have for $r > r_0$ (i.e. for outside the cylinder) the field equations,

$R_{ij} = 0$

A well-known solution of Einstein equations for empty space with cylindrical symmetric is that given by Levicivita [6] which is given as

(3.6) $ds^2 = -A^2 r^{-2c(1-c)}(dr^2 + dz^2) - r^2(1-c).\ d\varphi^2 + r^{2c}dt^2$

Where C and A being constants.

We use Licknerowicz boundary conditions namely that the metric potentials are *C* 'Continuous across the surface $r = r_0$. Thus the continuity of α, α', and β, β' gives us

(3.7) $B_1 = \frac{C}{r_0^2}$, $B_2 = C\log r_0 - \frac{C}{2}$

$C_1 = 4C\frac{r_0^{a-1}}{(a+1)(a+3)}[Ca + C - a - 3]$

$C_2 = \log A + C^2\ \log r_0 + \frac{(1-a)C^2}{4(a+3)} + \frac{2C(4a-2ac-2a+3+a^2)}{(1-a)^2(a+3)}$

Thus we have for the interior of the cylinder the solution

(3.8) $\alpha = \frac{2C\epsilon^2}{(C+3)(1+a)}[a + 3 + 2\epsilon^{-(1+a)}] - \frac{(1-a)C^2(\epsilon^4-1)}{(a+3)(1-a^2)} + \log A + C^2\log r_0 + \frac{2C(C^2+4C-2aC-2a+3)}{(1-a^2)(a+3)}$

(3.9) $\beta = \frac{C}{2}(\epsilon^2 + 2\log r_0 - 1)$

Where $\epsilon = \frac{r}{r_0}$

Also pressure and density are given by

(3.10) $8\pi\rho = 16\pi^2G^2e^{-2\alpha} + 4aCX[\frac{1}{1+a} - \frac{Cr^2}{a+3} + \frac{\epsilon^{-(1+a)}(aC+C-a-3)}{(1+a)(a+3)}]$

(3.11) $8\pi P_r = 16\pi^2G^2e^{-2\alpha} + X[\frac{4a}{1+a} + \frac{4a(1-a)\epsilon^{-(1+a)}}{(a+3)(1+a)} - \frac{(1-a)C^2\epsilon^2}{(a+3)} + C^2\epsilon^2]$

(3.12) $8\pi P_z = 16\pi^2G^2e^{-2\alpha} - X[\frac{4a}{1+a} + \frac{4a(1-a)\epsilon^{-(1+a)}}{(a+3)(1+a)} - \frac{(1-a)C^2\epsilon^2}{(a+3)} + C^2\epsilon^2]$

(3.13) $8\pi P_\Phi = 16\pi^2G^2e^{-2\alpha} - X[\frac{4a(aC+C-a-3)}{(a+1)(a+3)}\epsilon^{-(1+a)} - \frac{4}{1+a} + \frac{3(1-a)Cr^2}{a+3} - C^2\epsilon^2]$

Where $X = \frac{(\beta-\alpha)e^2}{r_0^2}$

IV. DISCUSSION/CONCLUSION

In General Relativity as given by Einstein there is no way of considering the spin effects on the geometry of space time. Further since a rotating system cannot be spherical. It is necessary to consider axi-symmetric distributions which are more physical. As such we have cons, idered the simplest axi–symmetric system namely cylindrically symmetric perfect fluid composed of particles having their spins aligned along the symmetry axis .To solve the field equations, we have used a suitable equation of state namely $\rho = ap_r$ and a judicious choice of one of the metric potentials. Further studies with rotating fluid distribution might give us more interesting aspects regarding the effects of the spin density.

REFERENCES

[1]. Hehl, F. W. Vander Heyde, P., Kerlick, C.D. and Nestar, J.M. (1976); Rev.
[2]. Hehl, F.W. (1974); Gen. Rel. Grav. 4,333.
[3]. Hehl, F. W. (1974); Gen.Rel. Grav. 5, 491.
[4]. Kerlick, G.D. (1975); Spin and torsion in gereral relativity foundations and implications
[5]. For Astrophysics and Cosmology, Ph.D. Thesis, Princeton University.
[6]. Kuchowicz, B. (1975); Acta Cosmologyca, 3, 109.
[7]. Levi –Civita, T. (1919); R.C. Accad. Lincel, 28, 101.
[8]. Prasanna, A.R. (1975); Phys. Rev. D11, 2076.
[9]. Prasanna, A.R. (1975); Phys. Rev. D11, 2083.
[10]. Singh, T. and Yadav R.B.S. (1978); Acta physics, Polon B8 837.
[11]. Skinner R. and Webb, I. (1977); Acta Phys. Polon, B8, 81.
[12]. Sah, A and Chandra, P (2016), Int. J Astronomy, Astropysics, 6, 494.
[13]. Yadav, R. B. S. et. al. (2020); B.H.U. Math. Soc. vol. 18, 99.
[14]. Yadav, R.B.S. and Kumar, J. (2020) Res. Rev. 5, 124-129.